1a Compute $\frac{3+2i}{2-3i}$ in standard form

Ans: $\frac{3+2i}{2-3i} = \frac{3+2i}{2-3i} \cdot \frac{2+3i}{2+3i}$

$= \frac{13i}{13} = i$

1b Compute $\log(-\sqrt{3}+i)$ in standard form by using principal branch.

Ans: $-\sqrt{3}+i = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2e^{\frac{5\pi}{6}i}$

$\therefore \log(-\sqrt{3}+i) = \log 2 + \frac{5\pi}{6} i$

1c Compute i^{-i} in standard form by using principal branch.

Ans: $\log i = \frac{\pi}{2} i$

$\Rightarrow i^{-i} = e^{-i\log i} = e^{\pi/2}$

2 Show that $f = \begin{cases} \frac{ze^{z}}{\bar{z}} & z \neq 0 \\ 1 & z = 0 \end{cases}$ is discontinuous at $z=0$

Ans: We let $z = iy$ with $y \in \mathbb{R}$ and $y \neq 0$.

$f(z) = \frac{iy\,e^{iy}}{-iy} = -e^{iy} = -(\cos y + i\sin y)$

$\lim_{z\to 0} f(z) = \lim_{y\to 0}(-\cos y - i\sin y) = -1 \neq f(0)$.

By definition of continuity, f is discontinuous at $z=0$.

3 (1) Show that $f = \begin{cases} \bar{z}^2 & z \neq 0 \\ 0 & z = 0 \end{cases}$ is diff. at and only at $z = 0$.

Ans: Suppose $z \in \mathbb{C}$ and $z \neq 0$,

$$f(z) = (x - iy)^2 = x^2 - y^2 - 2xyi = u + iv$$

$$u_x = 2x, \quad v_y = -2x$$
$$u_y = -2y, \quad v_x = -2y \Rightarrow u_x, u_y, v_x, v_y \in C$$

Thus the Cauchy-Riemann equation can not be satisfied for $z \neq 0$. f is only diff at $z = 0$.

(2)(i) Show that $\sin z = \sin x \cosh y + i \cos x \sinh y$ for $z = x + iy$

Ans: $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$$= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right)$$

$$= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)$$

$$= \frac{1}{2}\left(e^{y} + e^{-y}\right)\sin x + \frac{i}{2}\left(e^{y} - e^{-y}\right)\cos x$$

$$= \sin x \cosh y + i \cos x \sinh y$$

(2) (ii) Show that $\sin z$ is diff on $\mathbb{C}$ and find $\frac{d}{dz}\sin z$

Ans: let $\sin z = u + iv$ and by (2) (i)

$u_x = \cos x \cosh y \qquad v_y = \cos x \cosh y$

$u_y = \sin x \sinh y \qquad v_x = -\sin x \sinh y$

The Cauchy-Riemann equation is satisfied $\forall z \in \mathbb{C}$.

~~Thus it is diff on $\mathbb{C}$. (u, v are C^1).~~

and u, v are C^1, thus it's diff on $\mathbb{C}$.

Since $f'(z) = u_x + i v_x$

$= \cos x \cosh y - i \sin x \sinh y$

$= \frac{\cos x (e^y + e^{-y})}{2} - \frac{i \sin x (e^y - e^{-y})}{2}$

$= \frac{1}{2}(e^{y-ix} + e^{-y+ix})$

$= \cos z$